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CONFORMAL CHANGES OF RIEMANNIAN METRICS

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0. Introduction

Let M be an n-dimensional differentiable connected Riemannian manifold with metric tensor g. Since we consider several Riemannian metrics on the same manifold M, we denote by (M, g) the Riemannian manifold M with metric tensor g. The Riemannian metric g defines, in the tangent space at each point of the manifold, the inner product $g(X, Y)$ of two vectors X and Y at the point and the angle θ between two vectors by $\cos\theta = g(X, Y)/[\sqrt{g(X, X)}\cdot\sqrt{g(Y, Y)}]$. Let there be given two metrics g and g^* on M. If the angles between two vectors with respect to g and g^* are always equal to each other at each point of the manifold, we say that g and g^* are *conformally related* or that g and g^* are *conformal* to each other. A necessary and sufficient condition that g and g^* of M be conformal to each other is that there exist a function ρ on M such that $g^* = e^{2\rho}g$. We call such a change of metric $g \to g^*$ a *conformal change* of Riemannian metric. Yamabe [21] proved

Theorem A. *For any Riemannian metric given on a compact C^∞ differentiable manifold of dimension $n \geq 3$, there always exists a Riemannian metric which is conformal to the given metric and whose slalar curvature is constant.*

So in the study of conformal properties of a compact M we can assume the scalar curvature of M to be constant.

In the above discussion, what has been changed is the Riemannian metric g at each point of the manifold M. We are now going to consider point transformations which induce a conformal change of metric of the manifold.

Let (M, g) and (M', g') be two Riemannian manifolds, and $f: M \to M'$ a diffeomorphism. Then $g^* = f^{-1}g'$ is a Riemannian metric on M. When g^* and g are conformally related, that is, when there exists a function ρ on M such that $g^* = e^{2\rho}g$, we call $f: (M, g) \to (M', g')$ a *conformal transformation*. In particular, if ρ = constant, then f is called a *homothetic transformation* or a *homothety*; if $\rho = 0$, then f is called an *isometric transformation* or an *isometry*.

The group of all conformal (homothetic or isometric) transformations of (M, g) on itself is called a *conformal transformation (a homothetic transformation or an isometry) group* and is denoted by $C(M)$ ($H(M)$ or $I(M)$). We

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denote the connected components of the identity of $C(M), H(M)$ and $I(M)$ by $C_0(M), H_0(M)$ and $I_0(M)$ respectively.

If a vector field v defines an infinitesimal conformal transformation, then v satisfies $\mathscr{L}_v g = 2\rho g$, where $\mathscr{L}_v$ denotes the Lie derivative with respect to v, and ρ is a function on M. v defines an infinitesimal homothetic transformation or an infinitesimal isometry according as ρ is a constant or zero.

Riemannian manifolds with constant scalar curvature admitting an infinitesimal non-isometric conformal transformation have been studied by Bishop [2], Goldberg [2], [3], [4], [5], [6], Hsiung [8], [9], [10], Kobayashi [4], [5], [6], Lichnerowicz [14], Nagano [15], [16], [26], Obata [17], [18], [19], [27], Sawaki [28] and Yano [22], [23], [24], [25], [26], [27], [28]. A typical result may be quoted as follows.

Theorem B (*Goldberg* [3], *Obata* [18], [19], *Yano* [23]). *Suppose that a compact Riemannian manifold M of dimension $n \geq 2$ with constant scalar curvature K admits an infinitesimal non-isometric conformal transformation v so that $\mathscr{L}_v g = 2\rho g, \rho \neq$ const. Then a necessary and sufficient condition for M to be isometric to a sphere is*

$$\int_M G_{ji}\rho^j\rho^i dV = 0 ,$$

where $G_{ji} = K_{ji} - (1/n)Kg_{ji}, \rho^h = \rho_i g^{ih}, \rho_i = \nabla_i\rho, K_{ji}$ is the Ricci tensor, and dV is the volume element of M.

It is now a well-known conjecture that a compact Riemannian manifold with constant scalar curvature admitting a one-parameter group of non-isometric conformal transformations is isometric to a sphere.

Riemannian manifolds with constant scarlar curvature admitting a non-homothetic conformal transformation have been studied by Barbance [1], Goldberg [7], Hsiung [11], Kurita [13], Liu [11], Obata [17] and Yano [7]. A typical result may be quoted as follows.

Theorem C (*Goldberg & Yano* [7]). *Let (M, g) be a compact Riemannian manifold with constant scalar curvature K and admitting a non-homothetic conformal change $g^* = e^{2\rho}g$ such that $K^* = K$. If*

$$\int_M u^{-n+1}G_{ji}u^j u^i dV \geq 0 ,$$

where $u = e^{-\rho}, u_i = \nabla_i u, u^h = u_i g^{ih}$, then (M, g) is isometric to a sphere.

The purpose of the present paper is to establish some theorems on infinitesimal conformal transformations and conformal changes of metric, and to generalize the results obtained in Goldberg and Yano [7].

In the sequal, we need the following two theorems.

Theorem D (*Obata* [18]). *If a complete Riemannian manifold M of dimension* $n \geq 2$ *admits a non-constant function* ρ *such that* $\nabla_j \nabla_i \rho = -c^2 \rho g_{ji}$, *where c is a positive constant, then M is isometric to a sphere of radius* $1/c$ *in* $(n+1)$*-dimensional Euclidean space.*

Theorem E (*Ishihara & Tashiro* [12], *Tashiro* [20]). *If a complete Riemannian manifold M of dimension* $n \geq 2$ *admits a non-constant function* ρ *such that* $\nabla_j \nabla_i \rho = (1/n) \Delta \rho g_{ji}$, *where* $\Delta \rho = g^{ji} \nabla_j \nabla_i \rho$, *then M is conformal to a sphere in* $(n+1)$*-dimensional Euclidean space.*

Throughout the present paper, we assume that the Riemannian manifold M under consideration is compact and orientable. If M is not orientable, we need only to take an orientable double covering of M.

1. General formulas for infinitesimal conformal transformations

By g_{ji}, $\{_j{}^h{}_i\}$, ∇_i, $K_{kji}{}^h$, K_{ji} and K, we denote, respectively, the metric tensor, the Christoffel symbols formed with g_{ji}, the operator of covariant differentiation with respect to $\{_j{}^h{}_i\}$, the curvature tensor, the Ricci tensor and the scalar curvature of M.

We put

$$G_{ji} = K_{ji} - \frac{1}{n} K g_{ji} , \tag{1.1}$$

$$Z_{kji}{}^h = K_{kji}{}^h - \frac{1}{n(n-1)} K(\delta_k^h g_{ji} - \delta_j^h g_{ki}) , \tag{1.2}$$

$$W_{kji}{}^h = a Z_{kji}{}^h + b(\delta_k^h G_{ji} - \delta_j^h G_{ki} + G_k{}^h g_{ji} - G_j{}^h g_{ki}) , \tag{1.3}$$

where a, b are constant and $G_k{}^h = G_{ki} g^{ih}$. The tensor G_{ji} (respectively $Z_{kji}{}^h$) measures the deviation of the manifold M from being an Einstein space (respectively a space of constant curvature), and both tensors satisfy

$$G_{ji} g^{ji} = 0 , \quad Z_{tji}{}^t = G_{ji} , \quad W_{tji}{}^t = \{a + (n-2)b\} G_{ji} . \tag{1.4}$$

If $a + (n-2)b = 0$, then

$$W_{kji}{}^h = a C_{kji}{}^h , \tag{1.5}$$

where $C_{kji}{}^h$ is Weyl's conformal curvature tensor. Using Bianchi's identity, we can check

$$\nabla^j G_{ji} = \frac{n-2}{2n} \nabla_i K , \tag{1.6}$$

where $\nabla^j = g^{ji} \nabla_i$.

1.1. Formulas for an infinitesimal conformal transformation

When v^h defines an infinitesimal conformal transformation, we have

$$\mathscr{L}_v g_{ji} = \nabla_j v_i + \nabla_i v_j = 2\rho g_{ji} , \tag{1.7}$$

where $\rho = (1/n)\nabla_i v^i$.

Equation (1.7) and a general formula (see Yano [22]) for Lie derivatives,

$$\mathscr{L}_v\{_j{}^h{}_i\} = \tfrac{1}{2}g^{ht}\{\nabla_j(\mathscr{L}_v g_{it}) + \nabla_i(\mathscr{L}_v g_{jt}) - \nabla_t(\mathscr{L}_v g_{ji})\} ,$$

give

$$\mathscr{L}_v\{_j{}^h{}_i\} = \delta_j^h \rho_i + \delta_i^h \rho_j - g_{ji}\rho^h . \tag{1.8}$$

Equation (1.8) and a general formula (see Yano [22]),

$$\mathscr{L}_v K_{kji}{}^h = \nabla_k(\mathscr{L}_v\{_j{}^h{}_i\}) - \nabla_j(\mathscr{L}_v\{_k{}^h{}_i\}) ,$$

give

$$\mathscr{L}_v K_{kji}{}^h = -\delta_k^h \nabla_j \rho_i + \delta_j^h \nabla_k \rho_i - \nabla_k \rho^h g_{ji} + \nabla_j \rho^h g_{ki} , \tag{1.9}$$

from which follow

$$\mathscr{L}_v K_{ji} = -(n-2)\nabla_j \rho_i - \Delta\rho g_{ji} , \tag{1.10}$$

$$\mathscr{L}_v K = -2(n-1)\Delta\rho - 2\rho K , \tag{1.11}$$

where

$$\Delta\rho = g^{ji}\nabla_j \nabla_i \rho . \tag{1.12}$$

From (1.9), (1.10) and (1.11) we have

$$\mathscr{L}_v G_{ji} = -(n-2)\left(\nabla_j \rho_i - \frac{1}{n}\Delta\rho g_{ji}\right), \tag{1.13}$$

$$\begin{aligned}\mathscr{L}_v Z_{kji}{}^h = &-\delta_k^h \nabla_j \rho_i + \delta_j^h \nabla_k \rho_i - \nabla_k \rho^h g_{ji} + \nabla_j \rho^h g_{ki} \\ &+ \frac{2}{n}\Delta\rho(\delta_k^h g_{ji} - \delta_j^h g_{ki}) ,\end{aligned} \tag{1.14}$$

$$\begin{aligned}\mathscr{L}_v W_{kji}{}^h = \{a + (n-2)b\}\{&-\delta_k^h \nabla_j \rho_i + \delta_j^h \nabla_k \rho_i \\ &- \nabla_k \rho^h g_{ji} + \nabla_j \rho^h g_{ki} + \frac{2}{n}\Delta\rho(\delta_k^h g_{ji} - \delta_j^h g_{ki})\} .\end{aligned} \tag{1.15}$$

From (1.13), (1.14), (1.15) and $\mathscr{L}_v g^{ih} = -2\rho g^{ih}$, we have

(1.16) $$\mathscr{L}_v(G_{ji}G^{ji}) = -2(n-2)G_{ji}\nabla^j\rho^i - 4\rho G_{ji}G^{ji}\,,$$

(1.17) $$\mathscr{L}_v(Z_{kjih}Z^{kjih}) = -8G_{ji}\nabla^j\rho^i - 4\rho Z_{kjih}Z^{kjih}\,,$$

(1.18) $$\mathscr{L}_v(W_{kjih}W^{kjih}) = \{a + (n-2)b\}(-8G_{ji}\nabla^j\rho^i - 4\rho W_{kjih}W^{kjih})\,.$$

1.2. Integral formulas for an infinitesimal conformal transformation

We now assume that the manifold M is compact and orientable, and let there be given a vector field v^h in M. By a straight forward computation of

$$\nabla^j\left[\left\{\nabla_j v_i + \nabla_i v_j - \frac{2}{n}(\nabla_t v^t)g_{ji}\right\}v^i\right]$$

and integration over M, we obtain

(1.19) $$\begin{aligned}&\int_M\left\{g^{ji}\nabla_j\nabla_i v^h + K_i{}^h v^i + \frac{n-2}{n}\nabla^h(\nabla_i v^i)\right\}v_h dV\\ &\quad + \frac{1}{2}\int_M\left\{\nabla^j v^i + \nabla^i v^j - \frac{2}{n}(\nabla_t v^t)g^{ji}\right\}\\ &\quad\quad\cdot\left\{\nabla_j v_i + \nabla_i v_j - \frac{2}{n}(\nabla_s v^s)g_{ji}\right\}dV = 0\,,\end{aligned}$$

where dV is the volume element of M.

If v^h is a gradient vector field $v^h = \nabla^h\rho$, then (1.19) becomes

(1.20) $$\begin{aligned}&\int_M\left\{g^{ji}\nabla_j\nabla_i\rho^h + K_i{}^h\rho^i + \frac{n-2}{n}\nabla^h(\Delta\rho)\right\}\rho_h dV\\ &\quad + 2\int_M\left\{\nabla^j\rho^i - \frac{1}{n}(\Delta\rho)g^{ji}\right\}\left\{\nabla_j\rho_i - \frac{1}{n}(\Delta\rho)g_{ji}\right\}dV = 0\,.\end{aligned}$$

Since we have

(1.20)′ $$g^{ji}\nabla_j\nabla_i\rho^h = K_i{}^h\rho^i + \nabla^h(\Delta\rho)\,,$$

(1.20) can be reduced to

(1.21) $$\begin{aligned}&\int_M\left(K_{ji}\rho^j\rho^i + \frac{n-1}{n}\rho^i\nabla_i\Delta\rho\right)dV\\ &\quad + \int_M\left\{\nabla^j\rho^i - \frac{1}{n}(\Delta\rho)g^{ji}\right\}\left\{\nabla_j\rho_i - \frac{1}{n}(\Delta\rho)g_{ji}\right\}dV = 0\,,\end{aligned}$$

or

$$\begin{aligned}(1.22)\qquad &\int_M \left\{K_{ji}\rho^j\rho^i - \frac{n-1}{n}(\Delta\rho)^2\right\}dV \\ &\quad + \int_M \left\{\nabla^j\rho^i - \frac{1}{n}(\Delta\rho)g^{ji}\right\}\left\{\nabla_j\rho_i - \frac{1}{n}(\Delta\rho)g_{ji}\right\}dV = 0\,.\end{aligned}$$

If a non-constant function ρ satisfies $\Delta\rho = k\rho$ with a constant k, k being necessarily negative, (1.22) becomes

$$\begin{aligned}(1.23)\qquad &\int_M \left(K_{ji}\rho^j\rho^i - \frac{n-1}{n}k^2\rho^2\right)dV \\ &\quad + \int_M \left(\nabla^j\rho^i - \frac{1}{n}k\rho g^{ji}\right)\left(\nabla_j\rho_i - \frac{1}{n}k\rho g_{ji}\right)dV = 0\,,\end{aligned}$$

or

$$\begin{aligned}(1.24)\qquad &\int_M \left(K_{ji} + \frac{n-1}{n}kg_{ji}\right)\rho^j\rho^i dV \\ &\quad + \int_M \left(\nabla^j\rho^i - \frac{1}{n}k\rho g^{ji}\right)\left(\nabla_j\rho_i - \frac{1}{n}k\rho g_{ji}\right)dV = 0\,,\end{aligned}$$

by virtue of

$$\int_M k^2\rho^2 dV + \int_M kg_{ji}\rho^j\rho^i dV = 0\,,$$

derived from

$$\tfrac{1}{2}\Delta\rho^2 = \rho\Delta\rho + g_{ji}\rho^j\rho^i = k\rho^2 + g_{ji}\rho^j\rho^i\,.$$

Integral formulas (1.19), (1.20), (1.21) and (1.22) are valid for an arbitrary vector field v^h and an arbitrary function ρ, while integral formulas (1.23) and (1.24) for a function ρ satisfying $\Delta\rho = k\rho$.

If a Riemannian manifold with $K =$ const. admits an infinitesimal conformal transformation v^h, then from (1.11) we have

$$(1.25)\qquad \Delta\rho = -\frac{1}{n-1}K\rho\,,$$

and consequently (1.24) becomes

$$\begin{aligned}(1.26)\qquad &\int_M G_{ji}\rho^j\rho^i dV \\ &\quad + \int_M \left(\nabla^j\rho^i + \frac{1}{n(n-1)}K\rho g^{ji}\right)\left(\nabla_j\rho_i + \frac{1}{n(n-1)}K\rho g_{ji}\right)dV = 0\,.\end{aligned}$$

On the other hand, since $\nabla^j G_{ji} = 0$, we have

$$\nabla^j(G_{ji}\rho\rho^i) = G_{ji}\rho^j\rho^i + \rho G_{ji}\nabla^j\rho^i .$$

By substituting (1.16) for $G_{ji}\nabla^j\rho^i$ in the above equation and integrating over M we obtain

$$\int_M G_{ji}\rho^j\rho^i dV = \frac{1}{2(n-2)}\int_M \{4\rho^2 G_{ji}G^{ji} + \rho\mathscr{L}_v(G_{ji}G^{ji})\}dV . \tag{1.27}$$

Similarly, substitution of (1.17) and (1.18) for $G_{ji}\nabla^j\rho^i$ gives, respectively,

$$\int_M G_{ji}\rho^j\rho^i dV = \frac{1}{8}\int_M \{4\rho^2 Z_{kjih}Z^{kjih} + \rho\mathscr{L}_v(Z_{kjih}Z^{kjih})\}dV , \tag{1.28}$$

$$\int_M G_{ji}\rho^j\rho^i dV = \frac{1}{8}\int_M \left\{4\rho^2 W_{kjih}W^{kjih} + \frac{1}{\{a+(n-2)b\}^2}\rho\mathscr{L}_v(W_{kjih}W^{kjih})\right\} dV , \tag{1.29}$$

for $a + (n-2) \neq 0$.

2. Theorems on infinitesimal conformal transformations

We denote by (C) the following condition:
(C): The Riemannian manifold M is compact with constant scalar curvature K and admits an infinitesimal non-isometric conformal transformation v^h so that $\mathscr{L}_v g_{ji} = 2\rho g_{ji}$, $\rho \neq$ constant.

Then, first of all, from (1.26) we have

Theorem 2.1 (*Obata* [19]). *Suppose that M of dimension $n \geq 2$ satisfies* (C). *Then*

$$\int_M G_{ji}\rho^j\rho^i dV \leq 0 , \tag{2.1}$$

equality holding if and only if

$$\nabla_j\rho_i + \frac{1}{n(n-1)}K\rho g_{ji} = 0 ,$$

that is, if and only if M is isometric to a sphere.

Theorem 2.2 (*Yano* [23]). *Suppose that M of dimension $n \geq 2$ satisfies* (C). *If*

$$\int_M G_{ji}\rho^j\rho^i dV \geq 0 , \tag{2.2}$$

then M is isometric to a sphere.

Theorem 2.3 (*Goldberg* [3], *Obata* [19], *Yano* [24]). *Suppose that M of dimension $n \geq 2$ satisfies* (C). *Then in order that M be isometric to a sphere, it is necessary and sufficient that*

$$\int_M G_{ji}\rho^j\rho^i dV = 0 . \tag{2.3}$$

Suppose that M of dimension $n \geq 2$ satisfies (C) and one of the following conditions:

$$\mathscr{L}_v(G_{ji}G^{ji}) = 0 , \tag{2.4}$$

$$4\rho G_{ji}G^{ji} + \mathscr{L}_v(G_{ji}G^{ji}) = 0 , \tag{2.5}$$

$$\mathscr{L}_v(G_{ji}G^{ji}) = k\rho G_{ji}G^{ji} \qquad (k \geq -4) , \tag{2.6}$$

$$\mathscr{L}_v(G_{ji}G^{ji}) = k\rho^{2t+1}G_{ji}G^{ji} \qquad (k > 0,\ t\text{: integer}) , \tag{2.7}$$

then we see from (1.27) that (2.2) is satisfied and consequently that M is isometric to a sphere. Conversely, if M is isometric to a sphere, then G_{ji} vanishes identically and all the conditions above are satisfied.

Suppose that M of dimension $n > 2$ satisfies (C) and one of the following conditions:

$$\mathscr{L}_v(Z_{kjih}Z^{kjih}) = 0 , \tag{2.8}$$

$$4\rho Z_{kjih}Z^{kjih} + \mathscr{L}_v(Z_{kjih}Z^{kjih}) = 0 , \tag{2.9}$$

$$\mathscr{L}_v(Z_{kjih}Z^{kjih}) = k\rho Z_{kjih}Z^{kjih} \qquad (k \geq -4) , \tag{2.10}$$

$$\mathscr{L}_v(Z_{kjih}Z^{kjih}) = k\rho^{2t+1}Z_{kjih}Z^{kjih} \qquad (k > 0,\ t\text{: integer}) , \tag{2.11}$$

then we see from (1.28) that (2.2) is satisfied and consequently that M is isometric to a sphere. Conversely, if M is isometric to a sphere, then Z_{kjih} vanishes identically and all the conditions above are satisfied.

Similarly, suppose that M of dimension $n > 2$ satisfies (C) and one of the following conditions:

$$\mathscr{L}_v(W_{kjih}W^{kjih}) = 0 , \tag{2.12}$$

$$4\rho W_{kjih}W^{kjih} + \frac{1}{\{a + (n-2)b\}^2}\mathscr{L}_v(W_{kjih}W^{kjih}) = 0 , \tag{2.13}$$

$$\mathscr{L}_v(W_{kjih}W^{kjih}) = \{a + (n-2)b\}^2 k\rho W_{kjih}W^{kjih} \qquad (k \geq -4) , \tag{2.14}$$

$$\mathscr{L}_v(W_{kjih}W^{kjih}) = \{a + (n-2)b\}^2 k\rho^{2t+1}W_{kjih}W^{kjih} \qquad (k > 0,\ t\text{: integer}) , \tag{2.15}$$

$a + (n-2)b$ being different from zero. Then we see from (1.29) that (2.2) is satisfied and consequently that M is isometric to a sphere. Conversely, if M is isometric to a sphere, then $W_{kji}{}^h$ vanishes identically and all the conditions above are satisfied. Thus we have

Theorem 2.4. *Suppose that M of dimension $n > 2$ satisfies* (C). *In order that M be isometric to a sphere, it is necessary and sufficient that one of the conditions* (2.4)–(2.15) *be satisfied.*

3. General formulas for conformal changes of metric

In this section, we consider a conformal change of metric

$$g^*_{ji} = e^{2\rho} g_{ji} . \tag{3.1}$$

When Ω is a quantity formed with g, we denote by Ω^* the similar quantity formed with g^*.

3.1. Formulas for conformal changes of metric

We have

$$K^*_{kji}{}^h = K_{kji}{}^h - \delta^h_k \rho_{ji} + \delta^h_j \rho_{ki} - \rho_k{}^h g_{ji} + \rho_j{}^h g_{ki} , \tag{3.2}$$

$$K^*_{ji} = K_{ji} - (n-2)\rho_{ji} - \rho_a{}^a g_{ji} , \tag{3.3}$$

$$e^{2\rho} K^* = K - 2(n-1)\rho_a{}^a , \tag{3.4}$$

where

$$\rho_i = \nabla_i \rho , \qquad \rho^h = \rho_i g^{ih} ,$$

$$\rho_{ji} = \nabla_j \rho_i - \rho_j \rho_i + \frac{1}{2}\rho_a \rho^a g_{ji} , \qquad \rho_j{}^h = \rho_{ji} g^{ih} , \tag{3.5}$$

$$\rho_a{}^a = \Delta\rho + \frac{n-2}{2}\rho_a \rho^a , \qquad \Delta\rho = g^{ji}\nabla_j \rho_i .$$

From (3.2), (3.3), (3.4) and the definitions of $G_{ji}, Z_{kji}{}^h, W_{kji}{}^h$ we find

$$G^*_{ji} = G_{ji} - (n-2)(\nabla_j \rho_i - \rho_j \rho_i) + \frac{n-2}{n}(\Delta\rho - \rho_a \rho^a) g_{ji} , \tag{3.6}$$

$$\begin{aligned} Z^*_{kji}{}^h = Z_{kji}{}^h &- \delta^h_k(\nabla_j \rho_i - \rho_j \rho_i) + \delta^h_j(\nabla_k \rho_i - \rho_k \rho_i) \\ &- (\nabla_k \rho^h - \rho_k \rho^h) g_{ji} + (\nabla_j \rho^h - \rho_j \rho^h) g_{ki} \\ &+ \frac{2}{n}(\Delta\rho - \rho_a \rho^a)(\delta^h_k g_{ji} - \delta^h_j g_{ki}) , \end{aligned} \tag{3.7}$$

$$
\begin{aligned}
W^*_{kji}{}^h = W_{kji}{}^h + \{a + (n-2)b\} \\
\cdot \Big\{ -\delta^h_k(\nabla_j\rho_i - \rho_j\rho_i) + \delta^h_j(\nabla_k\rho_i - \rho_k\rho_i) \\
- (\nabla_k\rho^h - \rho_k\rho^h)g_{ji} + (\nabla_j\rho^h - \rho_j\rho^h)g_{ki} \\
+ \frac{2}{n}(\Delta\rho - \rho_a\rho^a)(\delta^h_k g_{ji} - \delta^h_j g_{ki}) \Big\} .
\end{aligned} \tag{3.8}
$$

If we put

$$u = e^{-\rho}, \qquad u_i = \nabla_i u, \tag{3.9}$$

then we have

$$\nabla_j u_i = -u(\nabla_j\rho_i - \rho_j\rho_i), \tag{3.10}$$

$$\Delta u = -u(\Delta\rho - \rho_a\rho^a), \tag{3.11}$$

and consequently, from (3.4), (3.6), (3.7) and (3.8),

$$K^* = u^2K + 2(n-1)u\Delta u - n(n-1)u_iu^i, \tag{3.12}$$

$$G^*_{ji} = G_{ji} + (n-2)P_{ji}, \tag{3.13}$$

$$Z^*_{kji}{}^h = Z_{kji}{}^h + Q_{kji}{}^h, \tag{3.14}$$

$$W^*_{kji}{}^h = W_{kji}{}^h + \{a + (n-2)b\}Q_{kji}{}^h, \tag{3.15}$$

where

$$P_{ji} = u^{-1}\Big(\nabla_j u_i - \frac{1}{n}\Delta u g_{ji}\Big), \qquad P_j{}^h = P_{ji}g^{ih}, \tag{3.16}$$

$$Q_{kji}{}^h = \delta^h_k P_{ji} - \delta^h_j P_{ki} + P_k{}^h g_{ji} - P_j{}^h g_{ki}. \tag{3.17}$$

From (3.16) and (3.17) we obtain

$$P_{ji}P^{ji} = u^{-2}\Big\{(\nabla^j u^i)(\nabla_j u_i) - \frac{1}{n}(\Delta u)^2\Big\}, \tag{3.18}$$

$$Q_{kjih}Q^{kjih} = 4(n-2)P_{ji}P^{ji}, \tag{3.12}$$

respectively. We also have, from (3.13), (3.14) and (3.15),

$$G^*_{ji}G^{*ji} = u^4\{G_{ji}G^{ji} + 2(n-2)G_{ji}P^{ji} + (n-2)^2P_{ji}P^{ji}\}, \tag{3.20}$$

$$Z^*_{kjih}Z^{*kjih} = u^4\{Z_{kjih}Z^{kjih} + 8G_{ji}P^{ji} + 4(n-2)P_{ji}P^{ji}\},\tag{3.21}$$

$$(3.22)\qquad W^*_{kjih}W^{*kjih} = u^4\{W_{kjih}W^{kjih} + 8(a + (n-2)b)^2 G_{ji}P^{ji} + 4(n-2)(a+(n-2)b)^2 P_{ji}P^{ji}\},$$

respectively. For the expression $G_{ji}P^{ji}$ in (3.20), (3.21) and (3.22), from (3.16) follows readily

$$(3.23)\qquad G_{ji}P^{ji} = u^{-1}G_{ji}\nabla^j u^i .$$

Proposition 3.1 ([17], [21]). *Suppose that K^* becomes a constant by a conformal change of metric. If K is nonpositive, then so is K^*.*

Proof. From (3.12) we have

$$K^*\int_M u^{-1}dV = \int_M uKdV - n(n-1)\int_M u^{-1}u_i u^i dV ,$$

and consequently, if $K \leq 0$, then $K^* \leq 0$.

Proposition 3.2. *Equation $K^* = u^2K$ never holds unless $u = const.$*

Proof. If $K^* = u^2K$ holds, then we have, from (3.12),

$$2u\Delta u - nu_i u^i = 0 ,$$

which implies

$$\int_M u^{-1}u_i u^i dV = 0 ,$$

and consequently $u_i = 0$, and $u = \text{const}$.

3.2. Integral formulas for a conformal change of metric

From (3.20) and (3.23) we can easily obtain

$$\int_M (u^{-3}G^*_{ji}G^{*ji} - uG_{ji}G^{ji})dV = (n-2)^2\left[-\int_M \frac{1}{n}u^i\nabla_i KdV + \int_M uP_{ji}P^{ji}dV\right]$$

by virtue of (1.6). Thus

$$(3.24)\qquad \int_M (u^{-3}G^*_{ji}G^{*ji} - uG_{ji}G^{ji})dV = (n-2)^2\left[\frac{1}{n}\int_M (\Delta u)KdV + \int_M uP_{ji}P^{ji}dV\right] .$$

Similarly, using (3.21) and (3.22) we can prove, respectively,

$$\begin{aligned}&\int_M (u^{-3}Z^*_{kjih}Z^{*kjih} - uZ_{kjih}Z^{kjih})dV \\ &\qquad = 4(n-2)\left[\frac{1}{n}\int_M (\Delta u)KdV + (n-2)\int_M uP_{ji}P^{ji}dV\right],\end{aligned} \tag{3.25}$$

$$\begin{aligned}&\int_M (u^{-3}W^*_{kjih}W^{*kjih} - uW_{kjih}W^{kjih})dV \\ &\qquad = 4(n-2)\{a+(n-2)b\}^2\left[\frac{1}{n}\int_M (\Delta u)KdV + \int_M uP_{ji}P^{ji}dV\right].\end{aligned} \tag{3.26}$$

From (3.20) we can easily obtain

$$\begin{aligned}&\int_M u^{-3}(G^*_{ji}G^{*ji} - G_{ji}G^{ji})dV = \int_M (u-u^{-3})G_{ji}G^{ji}dV \\ &\qquad + (n-2)^2\left[\frac{1}{n}\int_M (\Delta u)KdV + \int_M uP_{ji}P^{ji}dV\right],\end{aligned} \tag{3.27}$$

by virtue of

$$\int_M G_{ji}\nabla^j u^i dV = \frac{n-2}{2n}\int_M (\Delta u)KdV .$$

Similarly, using (3.21) and (3.22), we obtain, respectively,

$$\begin{aligned}&\int_M u^{-3}(Z^*_{kjih}Z^{*kjih} - Z_{kjih}Z^{kjih})dV \\ &\qquad = \int_M (u-u^{-3})Z_{kjih}Z^{kjih}dV \\ &\qquad\quad + 4(n-2)\left[\frac{1}{n}\int_M (\Delta u)KdV + \int_M uP_{ji}P^{ji}dV\right],\end{aligned} \tag{3.28}$$

$$\begin{aligned}&\int_M u^{-3}(W^*_{kjih}W^{*kjih} - W_{kjih}W^{kjih})dV \\ &\qquad = \int_M (u-u^{-3})W_{kjih}W^{kjih}dV \\ &\qquad\quad + 4(n-2)\{a+(n-2)b\}^2\left[\frac{1}{n}\int_M (\Delta u)KdV + \int_M uP_{ji}P^{ji}dV\right].\end{aligned} \tag{3.29}$$

Proposition 3.3. *If $K^* = K$ and $\mathscr{L}_{du}K = 0$, where $\mathscr{L}_{du}$ denotes the Lie derivative with respect to u^h, then, for an arbitrary integer p,*

$$\begin{aligned}
&\int_M u^{p-1}G_{ji}u^j u^i dV + \int_M u^{p+1}P_{ji}P^{ji}dV \\
&\quad = -(n+p-2)\Big[\int_M u^{p-2}(\nabla_j u_i)u^j u^i dV \\
&\qquad + \frac{1}{2n(n-1)}\int_M (u^{p-1} - u^{p-3})Ku_i u^i dV \\
&\qquad + \frac{1}{2}\int_M u^{p-3}(u_i u^i)dV\Big] .
\end{aligned} \tag{3.30}$$

In particular, if $p = 2 - n$, then

$$\int_M u^{-n+1}G_{ji}u^j u^i dV + \int_M u^{-n+3}p_{ji}P^{ji}dV = 0 . \tag{3.31}$$

Proof. From (3.18), by integration, directly computing $\nabla_j(u^{p-1}u_i\nabla^j u^i)$ and $\nabla_i(u^{p-1}u^i\Delta u)$, and using (1.20)′, which is true for any scalar function ρ, we easily obtain

$$\begin{aligned}
\int_M u^{p+1}P_{ji}P^{ji}dV &= -(p-1)\int_M u^{p-2}(\nabla_j u_i)u^j u^i dV \\
&\quad - \int_M u^{p-1}K_{ji}u^i u^i dV - \frac{n-1}{n}\int_M u^{p-1}u^i\nabla_i\Delta u dV \\
&\quad + \frac{p-1}{n}\int_M u^{p-2}u_i u^i \Delta u dV .
\end{aligned}$$

Substituting

$$\Delta u = \frac{1}{2(n-1)}(u^{-1} - u)K + \frac{1}{2}nu^{-1}u_i u^i ,$$

obtained from (3.12), in the above equation and using (1.1) an elementary computation leads readily to the required formula (3.30).

Proposition 3.4. *If $K^* = K$ and $\mathscr{L}_{du}K = 0$, then*

$$\int_M u^{-n+1}G_{ji}u^j u^i dV + \frac{1}{4(n-2)}\int_M u^{-n+3}Q_{kjih}Q^{kjih}dV = 0 . \tag{3.32}$$

Proof. From (3.19) and (3.31), we obtain (3.32).

Proposition 3.5. *If $\mathscr{L}_{du}K = 0$ and $G^*_{ji}G^{*ji} = G_{ji}G^{ji}$, then, for an arbitrary integer p,*

$$
(3.33)\quad \int_M (u^{p+1} - u^{p-3}) G_{ji} G^{ji} dV - 2(n-2)p \int_M u^{p-1} G_{ji} u^j u^i dV + (n-2)^2 \int_M u^{p+1} P_{ji} P^{ji} dV = 0 .
$$

In particular, if $p = 2 - n$, then

$$
(3.34)\quad \int_M (u^{-n+3} - u^{-n-1}) G_{ji} G^{ji} dV + 2(n-2)^2 \int_M u^{-n+1} G_{ji} u^j u^i dV + (n-2)^2 \int_M u^{-n+3} P_{ji} P^{ji} dV = 0 .
$$

Proof. From (3.20) and (3.23), by integration, directly computing $\nabla^j(u^p G_{ji} u^i)$ and using

$$
(\nabla^j G_{ji}) u^i = \frac{n-2}{2n} u^i \nabla_i K = \frac{n-2}{2n} \mathscr{L}_{du} K = 0 ,
$$

we can easily obtain the required formula (3.33).

Proposition 3.6. *If $\mathscr{L}_{du} K = 0$ and $Z^*_{kjih} Z^{*kjih} = Z_{kjih} Z^{kjih}$, then, for an arbitrary integer p,*

$$
(3.35)\quad \int_M (u^{p+1} - u^{p-3}) Z_{kjih} Z^{kjih} dV - 8p \int_M u^{p+1} G_{ji} u^j u^i dV + 4(n-2) \int_M u^{p+1} P_{ji} P^{ji} dV = 0 .
$$

In particular, if $p = 2 - n$, then

$$
(3.36)\quad \int_M (u^{-n+3} - u^{-n-1}) Z_{kjih} Z^{kjih} dV + 8(n-2) \int_M u^{-n+1} G_{ji} u^j u^i dV + 4(n-2) \int_M u^{-n+1} P_{ji} P^{ji} dV = 0 .
$$

Proof. (3.35) follows immediately from (3.21) and (3.23) in the same way as in the proof of Proposition 3.5.

Proposition 3.7. *If $\mathscr{L}_{du} K = 0$, $W^*_{kjih} W^{*kjih} = W_{kjih} W^{kjih}$ and*

$$
a + (n-2)b \neq 0 ,
$$

then, for an arbitrary integer p,

$$\begin{aligned}
(3.37) \qquad & \int_M (u^{p+1} - u^{p-3}) W_{kjih} W^{kjih} dV \\
& \quad - 8\{a + (n-2)b\}^2 p \int_M u^{p-1} G_{ji} u^j u^i dV \\
& \quad + 4(n-2)\{a + (n-2)b\}^2 \int_M u^{p+1} P_{ji} P^{ji} dV = 0 .
\end{aligned}$$

In particular, if $p = 2 - n$, then

$$\begin{aligned}
(3.38) \qquad & \int_M (u^{-n+3} - u^{-n-1}) W_{kjih} W^{kjih} dV \\
& \quad + 8(n-2)\{a + (n-2)b\}^2 \int_M u^{-n+1} G_{ji} u^j u^i dV \\
& \quad + 4(n-2)\{a + (n-2)b\}^2 \int_M u^{-n+3} P_{ji} P^{ji} dV = 0 .
\end{aligned}$$

Proof. (3.37) follows immediately from (3.22) and (3.23) in the same way as in the proof of Proposition 3.5.

4. Lemmas

Lemma 4.1. *Let F be a C^∞ function on a compact Riemannian manifold M such that*

$$\int_M F dV \leq 0 ,$$

and f be a C^∞ function such that

$$\begin{aligned}
c \leq f & \quad \text{in the domain} \quad F \leq 0 , \\
0 \leq f \leq c & \quad \text{in the domain} \quad F \geq 0 ,
\end{aligned}$$

where c is a positive constant. Then

$$\int_M fF dV \leq 0 .$$

Proof.

$$\begin{aligned}
\int_M fF dV &= \int_{F\leq 0} fF dV + \int_{F \geq 0} fF dV \\
&\leq c \int_{F \leq 0} F dV + c \int_{F \geq 0} F dV = c \int_M F dV \leq 0 .
\end{aligned}$$

Lemma 4.2. *If* $\int_M (\Delta u)KdV = 0$ *or* $\int_M \mathcal{L}_{du}KdV = 0$, *and* $G^*_{ji}G^{*ji} = G_{ji}G^{ji}$, *then, for an arbitrary non-positive* p,

$$\int_M (u^{p+1} - u^{p-3})G_{ji}G^{ji}dV \leq 0 . \tag{4.1}$$

In particular, if $p = 2 - n$, *then*

$$\int_M (u^{-n+3} - u^{-n-1})G_{ji}G^{ji}dV \leq 0 . \tag{4.2}$$

Proof. Now (3.27) implies

$$\int_M (u - u^{-3})G_{ji}G^{ji}dV \leq 0 .$$

Thus, if we put $F = (u - u^{-3})G_{ji}G^{ji}$, $f = u^p$, then the assumptions in Lemma 4.1 are satisfied, and consequently we have (4.1).

Similarly, we can prove

Lemma 4.3. *If* $\int_M (\Delta u)KdV = 0$ *or* $\int_M \mathcal{L}_{du}KdV = 0$, *and* $Z^*_{kjih}Z^{*kjih} = Z_{kjih}Z^{kjih}$, *then*

$$\int_M (u^{-n+3} - u^{-n-1})Z_{kjih}Z^{kjih}dV \leq 0 . \tag{4.3}$$

Lemma 4.4. *If* $\int_M (\Delta u)KdV = 0$ *or* $\int_M \mathcal{L}_{du}KdV = 0$, *and* $W^*_{kjih}W^{*kjih} = W_{kjih}W^{kjih}$, $a + (n-2)b \neq 0$, *then*

$$\int_M (u^{-n+3} - u^{-n-1})W_{kjih}W^{kjih}dV \leq 0 . \tag{4.4}$$

Lemma 4.5. *If* $K^* = K$, $\mathcal{L}_{du}K = 0$, *then*

$$\int_M u^{-n+1}G_{ji}u^ju^idV \leq 0 ,$$

equality holding if and only if

$$\nabla_j u_i - \frac{1}{n}\Delta u g_{ji} = 0 . \tag{4.5}$$

Proof. The lemma follows immediately from (3.31) and (3.16).

Lemma 4.6. *If* $K^* = K, \mathscr{L}_{du}K = 0$, *and*

$$\int_M u^{-n+1} G_{ji} u^j u^i dV \geq 0 , \tag{4.6}$$

then (4.5) *holds.*

Proof. Lemma 4.5 and the assumptions give the proof.

Lemma 4.7. *If* $K^* = K, \mathscr{L}_{du}K = 0$, *and* $G^*_{ji}G^{*ji} = G_{ji}G^{ji}$, *then* (4.5) *holds.*

Proof. (3.31), (3.34) and (4.2) imply (4.6), and hence (4.5) holds by Lemma 4.6.

Lemma 4.8. *If* $K^* = K, \mathscr{L}_{du}K = 0$ *and* $Z^*_{kjih}Z^{*kjih} = Z_{kjih}Z^{kjih}$, *then* (4.5) *holds.*

Proof. (3.31), (3.36) and (4.3) imply (4.6), and hence (4.5) holds by Lemma 4.6.

Lemma 4.9. *If* $K^* = K, \mathscr{L}_{du}K = 0, W^*_{kjih}W^{*kjih} = W_{kjih}W^{kjih}$, *and*

$$a + (n-2)b \neq 0 ,$$

then (4.5) *holds*

Proof. (3.31), (3.38) and (4.4) imply (4.6), and hence (4.5) holds by Lemma 4.6.

Lemma 4.10. *If* $\mathscr{L}_{du}K=0$, *and* (4.5) *holds for a non-constant function* u, *then* M *is isometric to a sphere.*

Proof. From (4.5), by an argument in the proof of Theorem E, it follows that the function u has exactly two critical points, P_+ and P_-, where u takes on the maximum and the minimum respectively. Then for each trajectory $\gamma(t)$ of the gradient of u we have $\lim_{t\to+\infty} \gamma(t) = P_+$ and $\lim_{t\to-\infty} \gamma(t) = P_-$.

Since $\mathscr{L}_{du}K = 0$, K is constant on each trajectory and hence on the whole M by continuity of K at P_+ and P_-. Then K must be positive [17]. Since M has positive constant scalar curvature, (4.5) implies $\nabla_j u_i + kug_{ji} = 0$, $k = K/n(n-1)$, [14], [27], and then, by Theorem D, M is isometric to a sphere.

5. Theorems on conformal changes of metric

Theorem 5.1. *If* M *of dimension* $n > 2$ *admits a conformal change of metric such that*

$$\int_M (\Delta u)KdV = 0 , \qquad G^*_{ji}G^{*ji} = u^4 G_{ji}G^{ji} ,$$

then M *is conformal to a sphere.*

Proof. (3.24) implies $P_{ji} = 0$ so that (4.5) holds by (3.16). Hence by Theorem E ([12], [20]) M is conformal to a sphere.

Theorem 5.2. *If M of dimension $n > 2$ with $K = const.$ admits a conformal change of metric such that $G^*_{ji}G^{*ji} = u^4 G_{ji}G^{ji}$, then M is isometric to a sphere.*

Proof. This is a consequence of Lemma 4.10 and Theorem 5.1.

Theorem 5.3. *If M of dimension $n > 2$ admits a conformal change of metric such that*

$$\int_M (\Delta u) K dV = 0 , \qquad Z^*_{kjih}Z^{*kjih} = u^4 Z_{kjih}Z^{kjih} ,$$

then M is conformal to a sphere.

Proof. The proof is the same as that of Theorem 5.1 except that (3.24) should be replaced by (3.25).

Theorem 5.4. *If M of dimension $n > 2$ with $K = const.$ admits a conformal change of metric such that $Z^*_{kjih}Z^{*kjih} = u^4 Z_{kjih}Z^{kjih}$, then M is isometric to a sphere.*

Proof. This is a consequence of Lemma 4.10 and Theorem 5.3.

Theorem 5.5. *If M of dimension $n > 2$ admits a conformal change of metric such that*

$$\int_M (\Delta u) K dV = 0 , \qquad W^*_{kjih}W^{*kjih} = u^4 W_{kjih}W^{kjih} ,$$
$$a + (n-2)b \neq 0 ,$$

then M is conformal to a sphere.

Proof. From (3.26) and the assumption of the theorem we have $P_{ji} = 0$, and consequently M is conformal to a sphere.

Theorem 5.6. *If M of dimension $n > 2$ with $K = const.$ admits a conformal change of metric such that $W^*_{kjih}W^{*kjih} = u^4 W_{kjih}W^{kjih}$, $a + (n-2)b \neq 0$, then M is isometric to a sphere.*

Proof. This is a consequence of Lemma 4.10 and Theorem 5.5.

Theorem 5.7. *If a compact M of dimension $n \geq 2$ admits a conformal change of metric such that $K^* = K$, $\mathscr{L}_{du}K = 0$, and (4.6) holds, then M is isometric to a sphere.*

Proof. (3.31) implies $P_{ji} = 0$, and consequently, by Lemma 4.10, M is isometric to a sphere.

Theorem 5.8. *If a compact M of dimension $n > 2$ admits a conformal change of metric such that $K^* = K$, $\mathscr{L}_{du}K = 0$, $G^*_{ji}G^{*ji} = G_{ji}G^{ji}$, then M is isometric to a sphere.*

Proof. By Lemma 4.7 and the assumption, we have $P_{ji} = 0$ and consequently by Lemma 4.10, M is isometric to a sphere.

Theorem 5.9. *If a compact M of dimension $n > 2$ admits a conformal change of metric such that*

$$K^* = K , \quad \mathscr{L}_{du}K = 0 , \quad Z^*_{kjih}Z^{*kjih} = Z_{kjih}Z^{kjih} ,$$

then M is isometric to a sphere.

Proof. By Lemma 4.8 and the assumptions, we have $P_{ji} = 0$ and consequently by Lemma 4.10, M is isometric to a sphere.

Theorem 5.10. *If a compact M of dimension $n > 2$ admits a conformal changes of metric such that*

$$K^* = K\,, \quad \mathscr{L}_{du}K = 0\,, \quad W^*_{kjih}W^{*kjih} = W_{kjih}W^{kjih}\,,$$
$$a + (n-2)b \neq 0\,,$$

then M is isometric to a sphere.

Proof. By Lemma 4.9 and the assumptions, we have $P_{ji} = 0$ and consequently, by Lemma 4.10, M is isometric to a sphere.

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